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Four-dimensional Wess–Zumino–Witten actions

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Abstract

We shall give an axiomatic construction of Wess–Zumino–Witten (WZW) actions valued in $G = \mathrm{SU}(N)$, $N \geq 3$. It is realized as a functor WZ from the category of conformally flat four-dimensional manifolds to the category of line bundles with connection that satisfies, besides the axioms of a topological field theory, the axioms which abstract the characteristics of WZW actions. To each conformally flat four-dimensional manifold Σ with boundary $\Gamma = \partial\Sigma$, a line bundle $L = \mathrm{WZ}(\Gamma)$ with connection over the space ΓG of mappings from Γ to G is associated. The WZW action is a non-vanishing horizontal section $\mathrm{WZ}(\Sigma)$ of the pullback bundle r^*L over ΣG by the boundary restriction $r : \Sigma G \to \Gamma G$. $\mathrm{WZ}(\Sigma)$ is required to satisfy a generalized Polyakov–Wiegmann formula with respect to the pointwise multiplication of the fields ΣG. Associated to the WZW action there is a geometric description of the extension of the Lie group $\Omega^3 G$ due to Mickelsson. In fact, we have two Abelian extensions of $\Omega^3 G$ that are in duality.

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1. Introduction

In this paper we shall give an axiomatic construction of the Wess–Zumino–Witten (WZW) action. Axiomatic approaches to field theories were introduced by Segal in two-dimensional conformal field theory (CFT), and by Atiyah [1,14] in topological field theory. The axioms

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abstract the functorial structure that the path integral would create if it existed as a mathematical object. Thus a CFT is defined as a Hilbert space representation of the operation of disjoint union and contraction on a category of manifolds with parameterized boundaries. The functional integral formalism was also explored by Gawedzki [6] to explain the WZW CFT. Singer [16] proposed a four-dimensional CFT in the language of Penrose's twistor space, where Riemann surfaces of two-dimensional CFT were replaced by conformally flat four-dimensional manifolds.

In a four-dimensional WZW model the space of field configurations is the space of all maps from closed four-dimensional manifolds with or without boundary into a compact Lie group. We know from the discussions in [16,19] that the geometric setting for CFT is most naturally given by the category of conformally flat manifolds. So we adopt this category of manifolds also for our WZW model. Let Σ be a conformally flat four-dimensional manifold with boundary $\Gamma = \partial\Sigma$ which may be the empty set. Let $G = \mathrm{SU}(N)$ with $N \geq 3$. The amplitude of the WZW model is given formally by the functional integration over fields $f \in \Sigma G = \mathrm{Map}(\Sigma, G)$ with the boundary restriction equal to the prescribed $g \in \Gamma G = \mathrm{Map}(\Gamma, G)$

$$A_\Sigma(g) = \int_{f\in\Sigma G;\, f|\Gamma=g} \exp\{2\pi \mathrm{i} S_\Sigma(f)\}\,\mathcal{D}f, \tag{1.1}$$

where $S_\Sigma(f)$ is defined by

$$S_\Sigma(f) = -\frac{\mathrm{i}k}{12\pi^2}\int_\Sigma \mathrm{tr}(\mathrm{d}f^{-1} \wedge *\mathrm{d}f) + C_\Sigma(f).$$

Since we deal with contributions that are topological in nature we omit the first term (kinetic term). The exponential of the second term

$$\mathrm{WZ}(\Sigma)(f) = \exp\{2\pi \mathrm{i} C_\Sigma(f)\} \tag{1.2}$$

is called the WZW action. (In [6,7] it is called an amplitude or a probability amplitude. In [3] it is called the WZW action.) When Σ has no boundary $C_\Sigma(f)$ is defined by

$$C_\Sigma(f) = \frac{\mathrm{i}}{240\pi^3}\int_{B^5} \mathrm{tr}(\mathrm{d}\tilde{f}\cdot\tilde{f}^{-1})^5, \tag{1.3}$$

where $\tilde{f}$ is an extension of f to a five-dimensional manifold B^5 with boundary $\partial B^5 = \Sigma$. Since Σ is a compact conformally flat manifold it is the boundary of a five-dimensional manifold B^5. But it is not clear that we can take such a smooth extension of f over B^5. If Σ is simply connected it is conformally equivalent to a four-dimensional sphere, and then, since $\pi_4(G) = 1$, there exists a smooth extension of f to the five-dimensional disc D^5 and $C_{S^4}(f)$ is defined up to $\mathbf{Z}$, that is, $\exp\{2\pi \mathrm{i} C_{S^4}(f)\}$ is well defined. The problem arises as to how to define the action $\mathrm{WZ}(\Sigma)(f)$ for general Σ without boundary. On the other hand, in (1.1) we are dealing with a four-manifold with boundary, so we must also give the definition of the action $\mathrm{WZ}(\Sigma)(f)$ for Σ with non-empty boundary. The above discussions lead to the following conclusion: *A four-dimensional WZW model means to assign a proper definition of the action* $\mathrm{WZ}(\Sigma)(f)$ *to every compact conformally flat four-manifold* Σ *with or without boundary.*

We shall construct the actions $\mathrm{WZ}(\Sigma)$ as the objects that satisfy several axioms. Our WZW actions are associated to four-dimensional manifolds with boundary and respect the functorial properties of various operations on the basic manifolds. Hence we impose on $\mathrm{WZ}(\Sigma)$ several axioms that are similar to those of topological field theories. Axioms of topological field theories were introduced by Atiyah [1]. They apply to a functor from the category of topological spaces to the category of vector spaces. Gawedzki [6] explored in the same spirit the axioms which characterize the amplitudes of two-dimensional WZW theory. Since our objects are not the amplitudes but the actions of the field, we describe our four-dimensional WZW theory as a functor WZ from the category of four-manifolds with boundary to the category of *complex line bundles*. This functor is required to satisfy the involutory axiom, the multiplicativity axiom and the associativity axiom that represent respectively the orientation reversal and the operations of disjoint union and contraction of the basic manifolds. Next we shall introduce two axioms that are characteristic of WZW models. We know that the action functional in field theory has topological effects, that is, it gives rise to the holonomy of a connection. So we require as our next axiom that the action $\mathrm{WZ}(\Sigma)$ gives rise to a four-dimensional analog of parallel transport associated to a connection of the complex line bundle. Higher-dimensional parallel transports as well as holonomies were discussed by Terashima [17], following the idea of Gawedzki [7] that relates isomorphism classes of line bundles with connection and the $U(1)$-holonomy coming from WZW action. The fundamental property of the WZW action is its behavior under the pointwise multiplication of fields. It is expressed by the Polyakov–Wiegmann formula [12], and its generalization to four-dimensional sphere was given by Mickelsson [10]. As our last axiom we demand that $\mathrm{WZ}(\Sigma)$ satisfies the generalized Polyakov–Wiegmann formula over ΣG. More precisely, the WZW actions can be stated as follows. A four-dimensional WZW model means a functor WZ that assigns to each manifold Σ, and its boundary $\Gamma = \partial\Sigma$, a line bundle $L = \mathrm{WZ}(\Gamma)$ over the space of maps ΓG, and a non-vanishing section $\mathrm{WZ}(\Sigma)$ over ΣG of the pullback line bundle r^*L by the boundary restriction map $r : \Sigma G \to \Gamma G$. The functor WZ satisfies the axioms of topological field theories. We demand that each line bundle $\mathrm{WZ}(\Gamma)$ has a connection and that $\mathrm{WZ}(\Sigma)$ is parallel with respect to the induced connection on r^*L. We impose moreover that on r^*L there is defined a product which is equivariant with respect to the product on ΣG through the Polyakov–Wiegmann formula

$$\mathrm{WZ}(\Sigma)(fg) = \mathrm{WZ}(\Sigma)(f) * \mathrm{WZ}(\Sigma)(g) \quad \text{for } f, g \in \Sigma G. \tag{1.4}$$

We shall see that $\mathrm{WZ}(\Sigma)$ is a positive integer for a compact Σ.

Here is a brief summary of each section. In Section 2, we explain following [16] that the category of conformally flat manifolds fits most naturally the construction of axiomatic CFT and our WZW model. In Section 2.2 we introduce the axioms of our WZW model. Gawedzki [6] gave two line bundles in duality over the loop space LG that correspond to the 2-cocycles obtained by transgressing the 3-curvature on G. In the same spirit we shall give in Section 3 two line bundles $\mathrm{WZ}(S^3)$ and $\mathrm{WZ}((S^3)')$ in duality over $\Omega_0^3 G$ that correspond to the 2-cocycles obtained by transgressing the 5-form over G. Here $\Omega_0^3 G$ is the space of smooth maps from S^3 to G that have degree 0. In fact, we have a 2-form on $\Omega_0^3 G$

$$\beta = \frac{\mathrm{i}}{240\pi^3} \int_{S^3} \mathrm{tr}(\mathrm{d}f \cdot f^{-1})^5, \tag{1.5}$$

which generates the integral cohomology class $H^2(\Omega_0^3 G, \mathbf{Z})$. Hence it defines a line bundle with connection on $\Omega_0^3 G$ with the curvature β. This is $\mathrm{WZ}(S^3)$. Let DG be the space of maps from a hemisphere D to G and let $D'G$ be the space of maps for the other hemisphere. We shall give a non-vanishing section $\mathrm{WZ}(D)$ of the pullback line bundle of $\mathrm{WZ}(S^3)$ by the boundary restriction map $r : DG \to \Omega_0^3 G$. Intuitively $\mathrm{WZ}(D)(f)$ is the holonomy associated to the curvature β over the four-dimensional path $f \in DG$. Similarly, we have a non-vanishing section $\mathrm{WZ}(D')$ of the pullback line bundle of $\mathrm{WZ}((S^3)')$ by $r' : D'G \to \Omega_0^3 G$. The connections on $\mathrm{WZ}(S^3)$ and $\mathrm{WZ}((S^3)')$ are given in Section 3.8, with respect to which $\mathrm{WZ}(D)$ and $\mathrm{WZ}(D')$ are parallel, respectively. In Section 4 we construct the functor WZ. The line bundle $\mathrm{WZ}(\Gamma)$ is defined as the tensor product of $\mathrm{WZ}(\Gamma_i)$ for each boundary component Γ_i parameterized by S^3, while each $\mathrm{WZ}(\Gamma_i)$ is defined as the pullback of $\mathrm{WZ}(S^3)$ or $\mathrm{WZ}((S^3)')$ by the map $\Gamma_i G \to S^3 G$ coming from the parameterization. The non-vanishing section $\mathrm{WZ}(\Sigma)$ of $r^* \mathrm{WZ}(\Gamma)$ is defined from the non-vanishing sections $\mathrm{WZ}(D)$ and $\mathrm{WZ}(D')$ by cutting and pasting methods and by using the dual relations, i.e. the associativity axiom. The connection on $\mathrm{WZ}(\Gamma)$ is induced from those on $\mathrm{WZ}(S^3)$ and $\mathrm{WZ}((S^3)')$ by a standard procedure. WZ satisfies the axioms that abstract the functorial structure of the WZW actions. In particular, we have the Polyakov–Wiegmann formula generalized to ΣG for any conformally flat four-manifold Σ. In Section 5 we shall discuss extensions of the Lie group $\Omega_0^3 G$. It is a well known observation that the two-dimensional WZW action gives a geometric description of central extensions of the loop group [2,6]. The $U(1)$-principal bundle over $\Omega_0^3 G$ associated to the line bundle $\mathrm{WZ}(S^3)$, however, does not have any group structure. Instead Mickelsson [10] gave an extension of $\Omega_0^3 G$ by the Abelian group $\mathrm{Map}(\mathcal{A}_3, U(1))$, where $\mathcal{A}_3$ is the space of connections on S^3. We shall explain two extensions of Mickelsson's type that are dual to each other.

2. Axioms for a four-dimensional WZW model

2.1. Category of conformally flat manifolds

The basic components of four-dimensional CFT are some well behaved class of four-dimensional manifolds M with parameterized boundaries, together with the natural operations of disjoint union

$$(M_1, M_2) \to M_1 \cup M_2,$$

and contraction

$$M \to \tilde{M},$$

where $\tilde{M}$ is obtained from M using the parameterization to attach a pair of boundary three-spheres to each other. A four-dimensional CFT is then defined as a Hilbert space representation of the operation of disjoint union and contraction on these basic components. Now we know that the geometric setting for this CFT is most naturally given by the conformal equivalence classes of conformally flat four-dimensional manifolds. This fact was explained by Singer [16], Zucchini [19] and Mickelsson and Scott [11].

Here we shall see following [16] the fact that the class of compact conformally flat four-dimensional manifolds with boundary is closed under the operation of sewing manifolds together across a boundary component. For any conformally flat M the developing map $M \to S^4$ is a well-defined conformal local diffeomorphism. A closed 3-manifold $N \subset M$ is called a *round* S^3 in M if it goes over diffeomorphically to a round S^3 in S^4 under development. This is well defined because the developing map is unique up to composition with conformal transformations. For standard M, the boundary ∂M consists of a disjoint union of round S^3s [13]. For each boundary component B one can find a neighborhood of B in M and a conformal diffeomorphism of this neighborhood onto a neighborhood of the equator in the northern hemisphere of S^4. If we have two boundary components B and $\tilde{B}$ of M and an orientation reversing conformal diffeomorphism $\psi : B \to \tilde{B}$, then B and $\tilde{B}$ can be attached using ψ and the resulting manifold will have a unique conformally flat structure compatible with the original one on M.

2.2. Four-dimensional WZW model

Now we give the precise definition of a four-dimensional WZW model.

Let $\mathcal{M}_4$ be the conformal equivalence classes of all compact conformally flat four-dimensional manifolds M with boundary $\partial M = \cup_{i\in I}\Gamma_i$ such that each oriented component Γ_i is a round S^3, and is endowed with a parameterization $p_i : S^3 \to \Gamma_i$. We distinguish positive and negative parameterizations $p_i : S^3 \to \Gamma_i, i \in I_\pm$, depending on whether p_i respects the orientation of Γ_i or not.

Let $\mathcal{M}$ be the category whose objects are three-dimensional manifolds Γ which are disjoint unions of round S^3's. A morphism between three-dimensional manifolds Γ_1 and Γ_2 is an oriented cobordism given by $\Sigma \in \mathcal{M}_4$ with boundary $\partial\Sigma = \Gamma_2 \cup (\Gamma_1')$, where the prime indicates the opposite orientation.

Let $\mathcal{L}$ be the category of complex line bundles.

Let $G = \mathrm{SU}(N)$, $N \geq 3$. In the following, the set of smooth mappings from a manifold M to G that are based at some point $p_0 \in M$ is denoted by $MG = \mathrm{Map}(M, G)$. MG becomes a group under product of mappings. For a $\Sigma \in \mathcal{M}_4$ with boundary $\Gamma = \partial\Sigma$, r denotes the restriction map

$$r : \Sigma G \to \Gamma G, \qquad r(f) = f|\Gamma. \tag{2.1}$$

A four-dimensional WZW model means a functor WZ from the category $\mathcal{M}$ to the category $\mathcal{L}$ which assigns:

(WZ1) to each manifold $\Gamma \in \mathcal{M}$, a complex line bundle $\mathrm{WZ}(\Gamma)$ over the space ΓG;

(WZ2) to each $\Sigma \in \mathcal{M}_4$, with $\partial\Sigma = \Gamma$, a non-vanishing section $\mathrm{WZ}(\Sigma)$ of the pullback line bundle $r^*\,\mathrm{WZ}(\Gamma)$.

Recall that the pullback bundle is by definition

$$r^*\,\mathrm{WZ}(\Gamma) = \{(f, u) \in \Sigma G \times \mathrm{WZ}(\Gamma), \pi u = r(f)\}, \tag{2.2}$$

and the section $\mathrm{WZ}(\Sigma)$ is given at $f \in \Sigma G$ by

$$\mathrm{WZ}(\Sigma)(f) = (f, u) \quad \text{with } u \in \pi^{-1}(r(f)) = \mathrm{WZ}(\Gamma)_{r(f)}.$$

WZ being a functor from $\mathcal{M}$ to $\mathcal{L}$, a conformal diffeomorphism $\alpha : \Gamma_1 \to \Gamma_2$ induces an isomorphism $\mathrm{WZ}(\alpha) : \mathrm{WZ}(\Gamma_1) \to \mathrm{WZ}(\Gamma_2)$ such that $\mathrm{WZ}(\beta\alpha) = \mathrm{WZ}(\beta)\mathrm{WZ}(\alpha)$ for $\beta : \Gamma_2 \to \Gamma_3$. Also if α extends to a conformal diffeomorphism $\Sigma_1 \to \Sigma_2$, with $\partial\Sigma_i = \Gamma_i$, $i = 1, 2$, then $\mathrm{WZ}(\alpha)$ takes $\mathrm{WZ}(\Sigma_1)$ to $\mathrm{WZ}(\Sigma_2)$.

The functor WZ satisfies the following axioms. A1–A3 represent in the category of line bundles the orientation reversal and the operation of disjoint union and contraction. These axioms are stated in the same manner as in topological field theories [1]. Axioms A4 and A5 are characteristic of the WZW model:

(A1) Involution:

$$\mathrm{WZ}(\Gamma') = \mathrm{WZ}(\Gamma)^*, \tag{2.3}$$

where $*$ indicates the dual line bundle.

(A2) Multiplicativity:

$$\mathrm{WZ}(\Gamma_1 \cup \Gamma_2) = \mathrm{WZ}(\Gamma_1) \otimes \mathrm{WZ}(\Gamma_2). \tag{2.4}$$

(A3) Associativity: For a composite cobordism $\Sigma = \Sigma_1 \cup_{\Gamma_3} \Sigma_2$ such that $\partial\Sigma_1 = \Gamma_1 \cup \Gamma_3$ and $\partial\Sigma_2 = \Gamma_2 \cup \Gamma_3'$, we have

$$\mathrm{WZ}(\Sigma)(f) = \langle \mathrm{WZ}(\Sigma_1)(f_1), \mathrm{WZ}(\Sigma_2)(f_2)\rangle \tag{2.5}$$

for any $f \in \Sigma G$, $f_i = f|\Sigma_i$, $i = 1, 2$, where $\langle\cdot,\cdot\rangle$ denotes the natural pairing

$$\mathrm{WZ}(\Gamma_1) \otimes \mathrm{WZ}(\Gamma_3) \otimes \mathrm{WZ}(\Gamma_3') \otimes \mathrm{WZ}(\Gamma_2) \to \mathrm{WZ}(\Gamma_1) \otimes \mathrm{WZ}(\Gamma_2). \tag{2.6}$$

More precisely, let $\mathrm{WZ}(\Sigma_1)(f_1) = (f_1, u_1 \otimes v)$ and $\mathrm{WZ}(\Sigma_2)(f_2) = (f_2, u_2 \otimes v')$ with $u_i \in \mathrm{WZ}(\Gamma_i)$ for $i = 1, 2$, and $v \in \mathrm{WZ}(\Gamma_3)$, $v' \in \mathrm{WZ}(\Gamma_3')$. From the definition $u_i \in \pi^{-1}(f_i|\Gamma_i)$, $v \in \pi^{-1}(f_1|\Gamma_3)$ and $v' \in \pi^{-1}(f_2|\Gamma_3')$. On the other hand, let $\mathrm{WZ}(\Sigma)(f) = (f, w_1 \otimes w_2) \in \mathrm{WZ}(\Gamma_1) \otimes \mathrm{WZ}(\Gamma_2)$ with $w_i \in \pi^{-1}(f|\Gamma_i)$ for $i = 1, 2$. Then axiom A3 says that $w_1 \otimes w_2 = \langle v', v\rangle u_1 \otimes u_2$. The multiplicative axiom A2 asserts that if $\partial\Sigma = \Gamma_2 \cup (\Gamma_1')$, then $\mathrm{WZ}(\Sigma)$ is a section of

$$r_1^* \mathrm{WZ}(\Gamma_1') \otimes r_2^* \mathrm{WZ}(\Gamma_2) = \mathrm{Hom}(r_1^* \mathrm{WZ}(\Gamma_1), r_2^* \mathrm{WZ}(\Gamma_2)). \tag{2.7}$$

Therefore any cobordism Σ between Γ_1 and Γ_2 induces a homomorphism of sections of pullback line bundles

$$\mathrm{WZ}(\Sigma) : C^\infty(\Sigma, r_1^* \mathrm{WZ}(\Gamma_1)) \to C^\infty(\Sigma, r_2^* \mathrm{WZ}(\Gamma_2)). \tag{2.8}$$

We impose:

1. $\mathrm{WZ}(\phi) = \mathrm{C}$ for ϕ the empty three-dimensional manifold, (2.9)
2. $\mathrm{WZ}(S^4) = 1$, (2.10)
3. $\mathrm{WZ}(\Gamma \times [0, 1]) = \mathrm{Id}(\mathrm{WZ}(\Gamma) \to \mathrm{WZ}(\Gamma))$. (2.11)

Corollary 2.1. *If Σ has no boundary* ($\partial\Sigma = \phi$), *then* $\mathrm{WZ}(\Sigma) \in \mathrm{C}$.

The following axioms are characteristic of WZW models:

(A4) For each $\Sigma \in \mathcal{M}_4$ with $\Gamma = \partial\Sigma$, $\mathrm{WZ}(\Gamma)$ has a connection, and $\mathrm{WZ}(\Sigma)$ is parallel with respect to the induced connection on $r^* \mathrm{WZ}(\Gamma)$.

(A5) Generalized Polyakov–Wiegmann formula: For each $\Sigma \in \mathcal{M}_4$ with $\Gamma = \partial\Sigma$, on the pullback line bundle $r^* \mathrm{WZ}(\Gamma)$ is defined a product $*$ with respect to which we have

$$\mathrm{WZ}(\Sigma)(fg) = \mathrm{WZ}(\Sigma)(f) * \mathrm{WZ}(\Sigma)(g) \quad \text{for any} \quad f, g \in \Sigma G. \tag{2.12}$$

The well-known Polyakov–Wiegmann formula extended by Mickelsson [10] is concerned with the case of the four-dimensional sphere, $\Sigma = S^4$.

From now on we shall construct the functor WZ step by step. In Section 3.5 we shall construct two line bundles over S^3G, which are $\mathrm{WZ}(S^3)$ and $\mathrm{WZ}((S^3)')$. In Section 4 we give the functor WZ of WZW actions step by step starting from $\mathrm{WZ}(S^3)$ and $\mathrm{WZ}((S^3)')$.

3. Line bundles on $\Omega^3 G$

3.1. $\Omega^3 G$

In the following, we denote by $\Omega^3 G$, instead of S^3G, the set of smooth mappings f from an S^3 to $G = \mathrm{SU}(N)$ that are based, i.e., $f(p_o) = 1$, at some point $p_o \in S^3$. It is known that $\Omega^3 G$ is not connected and is divided into denumerable sectors labeled by the soliton number (the mapping degree). Here we follow the explanation due to Singer [15] of these facts, see also [4,9]. Let the evaluation map, $\mathrm{ev} : S^3 \times \Omega^3 G \to G$, be defined by $\mathrm{ev}(m, \varphi) = \varphi(m), m \in S^3, \varphi \in \Omega^3 G$. The Maurer–Cartan form $g^{-1}\,\mathrm{d}g$ on G gives the identification of the tangent space T_eG at $e \in G$ and $\mathrm{Lie}\, G = \mathrm{su}(N)$. The primitive generators of the cohomology $H^*(G, R)$ are given by

$$\omega_3 = -\frac{1}{4\pi^2}\,\mathrm{tr}(g^{-1}\,\mathrm{d}g)^3, \qquad \omega_5 = -\frac{\mathrm{i}}{2\pi^2}\,\mathrm{tr}(g^{-1}\,\mathrm{d}g)^5, \ldots. \tag{3.1}$$

Integration on S^3 of the pullback of ω_{2k-1} by the evaluation map ev gives us the following $2(k-2)$ form on $\Omega^3 G$:

$$\nu_{2k-1} = \left(\frac{1}{2\pi\mathrm{i}}\right)^k \frac{((k-1)!)^2}{(2k-1)!} \int_{S^3} \mathrm{tr}(\mathrm{d}\varphi\,\varphi^{-1})^{2k-1}, \quad 3 \le 2k-1 \le 2N-1. \tag{3.2}$$

In particular, ν_3 is the mapping degree of φ

$$\deg \varphi = \frac{\mathrm{i}}{24\pi^2} \int_{S^3} \mathrm{tr}(\mathrm{d}\varphi\,\varphi^{-1})^3. \tag{3.3}$$

Proposition 3.1.

1. $S^3\,\mathit{Lie}\,G \overset{\exp}{\to} \Omega^3 G \overset{\deg}{\to} \mathrm{Z} \to 0$ *is exact.*
2. $\deg \varphi_1 \cdot \varphi_2 = \deg \varphi_1 + \deg \varphi_2$,

see [4,9].

3.2. 2-cocycle on S^4G

Let P_G be a G-principal bundle over S^4. Let $\mathcal{A}$ be the space of connections on P_G that are Lie G-valued 1-forms on P_G. Let $\mathcal{G} = S^4G$ be the group of based gauge transformations. The action of $\mathcal{G}$ on $\mathcal{A}$ is given by $A_g = g^{-1}Ag + g^{-1}\,\mathrm{d}g$ for $A \in \mathcal{A}$ and $g \in \mathcal{G}$. $F = F(A) = \mathrm{d}A + A^2$ denotes the curvature 2-form of A.

The Chern–Simons form on P_G is

$$\omega_5^0(A) = \mathrm{tr}(AF^2 - \tfrac{1}{2}A^3F + \tfrac{1}{10}A^5). \tag{3.4}$$

We have then $\mathrm{tr}(F^3) = \mathrm{d}\omega_5^0(A)$.

From Zumino [20] we know the relation

$$\omega_5^0(A_g) - \omega_5^0(A) = \mathrm{d}\alpha_4(A; g) + \tfrac{1}{10}\,\mathrm{tr}(\mathrm{d}g \cdot g^{-1})^5$$

with

$$\alpha_4(A; g) = \mathrm{tr}[-\tfrac{1}{2}V(AF + FA - A^3) + \tfrac{1}{4}(VA)^2 + \tfrac{1}{2}V^3A], \tag{3.5}$$

where $V = \mathrm{d}g \cdot g^{-1}$.

Let D^5 be a five-dimensional disc with boundary $\partial D^5 = S^4$. Integration over D^5 gives us the *gauge anomaly*

$$\Gamma(A, g) = \frac{\mathrm{i}}{48\pi^3}\int_{S^4}\mathrm{tr}[-V(AF + FA - A^3) + \tfrac{1}{2}(VA)^2 + V^3A] + C_5(g),$$

$$C_5(g) = \frac{\mathrm{i}}{240\pi^3}\int_{D^5}\mathrm{tr}(\mathrm{d}g \cdot g^{-1})^5, \tag{3.6}$$

here $g \in S^4G$ is extended to D^5G, in fact, we have such an extension by virtue of $\pi_4(G) = 1$. $C_5(g)$ may depend on the extension but it can be shown that the difference of two extensions is an integer, and $\exp(2\pi\mathrm{i}C_5(g))$ is independent of the extension.

We put, for $f, g \in S^4G$,

$$\gamma(f, g) = \frac{\mathrm{i}}{24\pi^3}\int_{S^4}\alpha_4(f^{-1}\,\mathrm{d}f, g) = \frac{\mathrm{i}}{48\pi^3}\int_{S^4}\mathrm{tr}[(\mathrm{d}g\,g^{-1})(f^{-1}\,\mathrm{d}f)^3 + \frac{1}{2}(\mathrm{d}g\,g^{-1}f^{-1}\,\mathrm{d}f)^2 + (\mathrm{d}g\,g^{-1})^3(f^{-1}\,\mathrm{d}f)]. \tag{3.7}$$

and

$$\omega(f, g) = \Gamma(f^{-1}\,\mathrm{d}f, g) = \gamma(f, g) + C_5(g). \tag{3.8}$$

Remark 3.1. Here we shall look at Mickelson's 2-cocycle for his Abelian extension of Ω^3G [10]. The cochain α_4 in (3.5) is a one-cochain on the group S^4G, valued in $\mathrm{Map}(\mathcal{A}_4, R)$. The coboundary $\delta\alpha_4$ is given by

$$\delta\alpha_4(A : g_1, g_2) = \mathrm{d}\beta + \alpha_4(g_1^{-1}\,\mathrm{d}g_1; g_2),$$

$$\beta(A; g_1, g_2) = -\mathrm{tr}[\tfrac{1}{2}(\mathrm{d}g_2\,g_2^{-1})(g_1^{-1}\,\mathrm{d}g_1)(g_1^{-1}Ag_1) - \tfrac{1}{2}(\mathrm{d}g_2\,g_2^{-1})(g_1^{-1}Ag_1)(g_1^{-1}\,\mathrm{d}g_1)].$$

Mickelson's 2-cocycle $\gamma_\Delta(A;\, f, g)$ is defined as the integration of $\delta\alpha_4(A;\, g_1, g_2)$ over any region $\Delta \subset S^4$

$$\gamma_\Delta(A;\, f, g) = \frac{\mathrm{i}}{24\pi^3}\int_\Delta \delta\alpha_4(A;\, f, g). \tag{3.9}$$

But for $\Delta = S^4$, it is independent of A and

$$\gamma_{S^4}(A;\, f, g) = \int_{S^4} \delta\alpha_4(A;\, f, g) = \int_{S^4} \alpha_4(f^{-1}\,\mathrm{d}f, g) = \gamma(f, g) \tag{3.10}$$

for $f, g \in S^4G$. Hence, instead of $\gamma_{S^4}(A;\, f, g)$, we use more simple $\gamma(f, g)$ for our purpose.

Remark 3.2. We have

$$\gamma(F, G) = \gamma_D(A;\, F, G) + \gamma_{D'}(A;\, F, G) \tag{3.11}$$

for any $A \in \mathcal{A}_4$. Here D is an oriented hemisphere of S^4 and D' is the other hemisphere: $D \cup D' = S^4$.

Lemma 3.1 (Polyakov–Wiegmann). *For $f, g \in S^4G$ we have*

$$C_5(fg) = C_5(f) + C_5(g) + \gamma(f, g) \bmod \mathbf{Z}. \tag{3.12}$$

The following formula was proved by Mickelsson [10, Lemma 4.3.7]:

$$C_5(fg) = C_5(f) + C_5(g) + \gamma_{S^4}(A;\, f, g) \bmod \mathbf{Z}.$$

Since $\gamma_{S^4}(A;\, f, g) = \gamma(f, g)$ from (3.10) we have the proposition.

3.3. *Line bundle WZ(ϕ)*

Now we are prepared to define the line bundle $\mathrm{WZ}(\phi)$ over $\mathrm{Map}(\partial S^4, G) = \phi$, and the section $\mathrm{WZ}(S^4)$ of the pullback line bundle of $\mathrm{WZ}(\phi)$ by the empty restriction map $r : S^4G \to \phi$.

Let L_ϕ be the quotient of $S^4G \times \mathrm{C}$ by the equivalence relation

$$(f, c) \sim (g, c\exp\{2\pi\mathrm{i}\omega(f, f^{-1}g)\}). \tag{3.13}$$

Then L_ϕ is a line bundle over $\mathrm{Map}(\partial S^4, G) = \phi$ with the transition function $\exp\{2\pi\mathrm{i}\omega(f, f^{-1}g)\}$, which we shall define as $\mathrm{WZ}(\phi)$. Recall that S^4G is contractible. We have then

$$\mathrm{WZ}(\phi) \simeq \mathrm{C}. \tag{3.14}$$

The isomorphism is given by

$$[f, c] \to c\exp\{-2\pi\mathrm{i}C_5(f)\}.$$

It is well defined because of the Polyakov–Wiegmann formula. Let $r^*\mathrm{WZ}(\phi)$ be the pullback line bundle of $\mathrm{WZ}(\phi)$ by the empty restriction map $r : S^4G \to \phi$. The section $\mathrm{WZ}(S^4)$ of $r^*\mathrm{WZ}(\phi)$ over any $f \in S^4G$ is given by

$$\mathrm{WZ}(S^4)(f) = [f, \exp\{2\pi\mathrm{i}C_5(f)\}] \in \mathrm{WZ}(\phi). \tag{3.15}$$

By the isomorphism of (3.14) we can also write

$$\mathrm{WZ}(S^4) = 1 \in \mathrm{C}.$$

We can define the product on the line bundle $\mathrm{WZ}(\phi) \simeq \mathrm{C}$ in an obvious way, but we shall look this product more precisely, rather superfluously, for the sake of later sections. In $S^4G \times \mathrm{C}$ we define the product by putting

$$(f, a) * (g, b) = (fg, ab \exp\{2\pi \mathrm{i}\gamma(f, g)\}). \tag{3.16}$$

Since the equivalence relation (3.13) respects the product, it gives a product on the line bundle $\mathrm{WZ}(\phi)$. The Polyakov–Wiegmann formula (3.12) is stated as follows:

$$\mathrm{WZ}(S^4)(fg) = \mathrm{WZ}(S^4)(f) * \mathrm{WZ}(S^4)(g) \quad \text{for} \quad f, g \in S^4G. \tag{3.17}$$

3.4. Notations and definitions

In this section, we shall prepare some notations, definitions and elementary properties that will be used in the following sections.

Let Ω^3G be as before the set of smooth mappings from S^3 to $G = \mathrm{SU}(N)$ that are based. Ω^3G is not connected but divided into the connected components by deg. We put

$$\Omega^3_0G = \{g \in \Omega^3G; \deg g = 0\}. \tag{3.18}$$

The oriented four-dimensional disc with boundary S^3 is denoted by D, while that with opposite orientation is denoted by D'. The composite cobordism of D and D' becomes S^4. We write as before $DG = \mathrm{Map}(D, G)$ and $D'G = \mathrm{Map}(D', G)$. The restriction to S^3 of an $f \in DG$ has degree 0; $f|S^3 \in \Omega^3_0G$.

For an $a \in \Omega^3_0G$ we denote by Da the set of those $g \in DG$ that is a smooth extension of a, respectively, $D'a$ is the set of those $g' \in D'G$ that is a smooth extension of a. For $f \in Da$ and $g \in Db$ one has $fg \in D(ab)$, and every element of $D(ab)$ is of this form. Similarly for $D'(ab)$. We denote by $g \vee g' \in S^4G$ the map obtained by sewing $g \in DG$ and $g' \in D'(g|S^3)$.

The prime will indicate that the function expressed by the letter is defined on D', for example, $1'$ is the constant function $D' \ni x \to 1'(x) = e \in G$, while 1 is the constant function $D \ni x \to 1(x) = e \in G$. We write

$$D'f = \{f' \in D'G : f'|S^3 = f|S^3\}, \qquad Df' = \{f \in DG : f|S^3 = f'|S^3\}.$$

Let $f, g \in DG$ and $f|S^3 = g|S^3$. From (3.7) and (3.8) we see that $\gamma(f \vee f', f^{-1}g \vee 1')$ and $\omega(f \vee f', f^{-1}g \vee 1')$ are independent of $f' \in Df$

$$\gamma(f \vee f', f^{-1}g \vee 1') = \frac{\mathrm{i}}{48\pi^3}\int_D \mathrm{tr}\left[(\mathrm{d}g\, g^{-1})(f^{-1}\,\mathrm{d}f)^3 + \frac{1}{2}(\mathrm{d}g\, g^{-1} f^{-1}\,\mathrm{d}f)^2 + (\mathrm{d}g\, g^{-1})^3(f^{-1}\,\mathrm{d}f)\right]. \tag{3.19}$$

Similarly, for $f', g' \in D'G$ such that $f'|S^3 = g'|S^3$. $\gamma(g \vee g', 1 \vee (g')^{-1}f')$ and $\omega(g \vee g', 1 \vee (g')^{-1}f')$ are independent of $g \in Dg'$. Hence $\exp\{2\pi\mathrm{i}\omega(f \vee \cdot, f^{-1}g \vee 1')\}$ and $\exp\{2\pi\mathrm{i}\omega(\cdot \vee f', 1 \vee (f')^{-1}g')\}$ are constants of $U(1)$.

Definition 3.1.

1. We put, for $f, g \in DG$ such that $f|S^3 = g|S^3$,

$$\chi(f, g) = \exp\{2\pi i\omega(f \vee \cdot, f^{-1}g \vee 1')\}. \tag{3.20}$$

2. We put, for $f', g' \in D'G$ such that $f'|S^3 = g'|S^3$,

$$\chi'(f', g') = \exp\{2\pi i\omega(\cdot \vee f', 1 \vee (f')^{-1}g')\}. \tag{3.21}$$

Lemma 3.2.

1. *For $f, g \in DG$ such that $f|S^3 = g|S^3$, we have $\chi(f, g) \in U(1)$ and*

$$\exp\{2\pi i C_5(g \vee f')\} = \exp\{2\pi i C_5(f \vee f')\}\chi(f, g) \quad \textit{for any} \quad f' \in D'f. \tag{3.22}$$

2. *For $f', g' \in D'G$ such that $f'|S^3 = g'|S^3$, we have $\chi'(f', g') \in U(1)$ and*

$$\exp\{2\pi i C_5(f \vee g')\} = \exp\{2\pi i C_5(f \vee f')\}\chi'(f', g') \quad \textit{for any} \quad f \in Df'. \tag{3.23}$$

The lemma follows from the Polyakov–Wiegmann formula.

3.5. Line bundles $WZ(S^3)$ and $WZ((S^3)')$

Now we shall give two line bundles on $\Omega_0^3 G$ that are dual to each other. We shall follow the arguments due to Gawedzki [6] that were developed to construct two line bundles in duality over the loop group LG and to give the definition of WZW action on a hemisphere.

We consider the following quotient:

$$L = D'G \times \mathrm{C}/ \sim' , \tag{3.24}$$

where $\sim'$ is the equivalence relation defined by

$$(f', c') \sim' (g', d') \quad \text{if and only if} \quad f'|S^3 = g'|S^3, \quad d' = c'\chi'(f', g'). \tag{3.25}$$

The equivalence class of (f', c') is denoted by $[f', c']$. We define the projection

$$\pi : L \to \Omega_0^3 G$$

by $\pi([f', c']) = f'|S^3$. L becomes a line bundle on $\Omega_0^3 G$ with the transition function $\chi'(f', g')$.

More precisely, let $a \in \Omega_0^3 G$ and take $f' \in D'a$. A coordinate neighborhood of a is given by

$$U_{f'} = \{g'|S^3;\ g' \in V_{f'}\},$$

$$V_{f'} = \{g' \in D'G,\ g' = \exp X \cdot f';\ X \in D'(\mathrm{Lie}\, G),\ \|X\| < \delta\}.$$

The local trivialization of L is given by the map $\pi^{-1}(U_{f'}) \ni [h', c'] \to (h'|S^3, c')$

$$\pi^{-1}(U_{f'}) \simeq U_{f'} \times \mathrm{C}.$$

The transition function $\chi_{U_{f'},U_{g'}}(b)$ of L at $b \in U_{f'} \cap U_{g'}$ becomes as follows. Let $b \in U_{f'} \cap U_{g'}$. Let $h' \in V_{f'}$ and $k' \in V_{g'}$ be such that $h'|S^3 = k'|S^3 = b$. For $\xi = [h', c'] = [k', d'] \in \pi^{-1}(b)$ we have obviously $d' = \chi'(h', k')c'$. Hence

$$\chi_{U_{f'},U_{g'}}(b) = \chi'(h', k'). \tag{3.26}$$

The line bundle L is what we wanted to construct and will be denoted by $\mathrm{WZ}(S^3)$.

In regard to the involution axiom A1 which $\mathrm{WZ}(\cdot)$ is required to satisfy we must define another line bundle on $\Omega_0^3 G$ corresponding to S^3 with opposite orientation. This line bundle $\mathrm{WZ}((S^3)')$ is defined by

$$\mathrm{WZ}((S^3)') = DG \times \mathrm{C}/\sim \tag{3.27}$$

with the equivalence relation

$$(f, c) \sim (g, d) \quad \text{if and only if} \quad f|S^3 = g|S^3, \quad \mathrm{d} = c\chi(f, g). \tag{3.28}$$

The projection $\pi : \mathrm{WZ}((S^3)') \to \Omega_0^3 G$ is given by $[f, c] \to f|S^3$. It is a line bundle with the transition function $\chi(f, g)$.

$\mathrm{WZ}(S^3)$ and $\mathrm{WZ}((S^3)')$ are in duality so that the involution axiom A1 is verified for these line bundles. In fact, the duality

$$\mathrm{WZ}(S^3) \times \mathrm{WZ}((S^3)') \to \mathrm{C}$$

is defined by

$$\langle [f', c'], [f, c] \rangle = cc' \exp\{-2\pi \mathrm{i} C_5(f \vee f')\}, \tag{3.29}$$

where $f|S^3 = f'|S^3 \in \Omega_0^3 G$. If we note the evident fact that $\gamma(F, 1 \vee h')$ (resp. $\gamma(F, h \vee 1')$) in (3.19) is given by an integration over D' (resp. D), we see that the product of transition rules (3.25) and (3.28) imply the transition rule (3.13) of $\mathrm{WZ}(\phi)$

$$\chi(f, g)\chi'(f', g') = \exp\{2\pi \mathrm{i}\omega(f \vee f', f^{-1}g \vee (f')^{-1}g')\}. \tag{3.30}$$

Hence

$$\mathrm{WZ}(S^3) \otimes \mathrm{WZ}((S^3)') = \mathrm{WZ}(\phi). \tag{3.31}$$

Composed with (3.14) this implies the above duality.

3.6. Non-vanishing sections WZ(D) and WZ(D′)

Let $r : DG \to S^3 G$ and $r' : D'G \to S^3 G$ be the restriction maps.

We put for $f \in DG$,

$$\mathrm{WZ}(D)(f) = [f', \exp\{2\pi \mathrm{i} C_5(f \vee f')\}] \in \mathrm{WZ}(S^3)|_{r(f)}. \tag{3.32}$$

Then we see from Lemma 3.2 that $\mathrm{WZ}(D)$ gives a non-vanishing section of the pullback line bundle $r^* \mathrm{WZ}(S^3)$.

In the same way we put for $f' \in D'G$,

$$\mathrm{WZ}(D')(f') = [f, \exp\{2\pi \mathrm{i} C_5(f \vee f')\}] \in \mathrm{WZ}((S^3)')|_{r'(f')}. \tag{3.33}$$

$\mathrm{WZ}(D')$ defines a non-vanishing section of $(r')^* \mathrm{WZ}((S^3)')$.

Proposition 3.2. *For $f \in DG$ and $f' \in D'G$ such that $f|S^3 = f'|S^3$*

$$\langle \mathrm{WZ}(D)(f), \mathrm{WZ}(D')(f') \rangle = \mathrm{WZ}(S^4)(f \vee f'). \tag{3.34}$$

In fact, both sides are equal to $\exp\{2\pi \mathrm{i} C_5(f \vee f')\}$.

3.7. Products in $r^*WZ(S^3)$ and $(r')^*WZ((S^3)')$

The total space of the pullback bundle $r^* \mathrm{WZ}(S^3)$ is written as

$$r^* \mathrm{WZ}(S^3) = \{(f, \lambda);\ f \in DG, \lambda = [f', c'] \in \mathrm{WZ}(S^3)_{r(f)}\}.$$

We define the product in $r^* \mathrm{WZ}(S^3)$ by the formula

$$(f, \lambda) * (g, \mu) = (fg, \nu), \tag{3.35}$$

where, for $\lambda = [f', a'] \in \mathrm{WZ}(S^3)_{r(f)}$ and $\mu = [g', b'] \in \mathrm{WZ}(S^3)_{r(g)}$, $\nu = [f'g', c'] \in \mathrm{WZ}(S^3)_{r(fg)}$ is defined by

$$c' = a'b' \exp\{2\pi \mathrm{i} \gamma(f \vee f', g \vee g')\}, \tag{3.36}$$

ν does not depend on the representations of λ and μ, and the product is well defined.

We have

$$\mathrm{WZ}(D)(fg) = \mathrm{WZ}(D)(f) * \mathrm{WZ}(D)(g) \quad \text{for} \quad f, g \in DG. \tag{3.37}$$

In fact, this follows from the definition:

$$\mathrm{WZ}(D)(f) = [f', \exp\{2\pi \mathrm{i} C_5(f \vee f')\}],$$

and the Polyakov–Wiegmann formula.

Similarly we have the product on $(r')^* \mathrm{WZ}((S^3)')$ over $D'G$. It is given by

$$(f', \alpha) * (g', \beta) = (f'g', \gamma), \tag{3.38}$$

where for $\alpha = [f, a] \in \mathrm{WZ}((S^3)')_{r'(f')}$ and $\beta = [g, b] \in \mathrm{WZ}((S^3)')_{r'(g')}$, $\gamma = [fg, c] \in \mathrm{WZ}((S^3)')_{r'(f'g')}$ is defined by

$$c = ab \exp\{2\pi \mathrm{i} \gamma(f \vee f', g \vee g')\}. \tag{3.39}$$

We have

$$\mathrm{WZ}(D')(f'g') = \mathrm{WZ}(D')(f') * \mathrm{WZ}(D')(g') \quad \text{for} \quad f', g' \in D'G. \tag{3.40}$$

We note that product operations on $r^* \mathrm{WZ}(S^3)$ and on $(r')^* \mathrm{WZ}((S^3)')$ are compatible with the duality

$$r^* \mathrm{WZ}(S^3) \times (r')^* \mathrm{WZ}((S^3)') \to \mathrm{WZ}(\phi) \simeq \mathrm{C}, \tag{3.41}$$

that is, for $(f, \lambda), (g, \mu) \in r^* \mathrm{WZ}(S^3)$ and for $(f', \lambda'), (g', \mu') \in (r')^* \mathrm{WZ}((S^3)')$ such that $r(f) = r'(f')$ and $r(g) = r'(g')$, we have

$$\langle (f, \lambda) * (g, \mu), (f', \lambda') * (g', \mu') \rangle = \langle \lambda, \lambda' \rangle * \langle \mu, \mu' \rangle, \tag{3.42}$$

the right-hand side being the product in $\mathrm{WZ}(\phi) \simeq \mathrm{C}$.

3.8. *Connections on* $\mathrm{WZ}(S^3)$ *and* $\mathrm{WZ}((S^3)')$

Next we define a connection on $\mathrm{WZ}(S^3)$. They are described as follows. Let $b \in \Omega_0^3 G$ and $U_{f'}$ be a coordinate neighborhood described in Section 3.5. On $U_{f'}$ we put

$$\theta_{U_{f'}}(b)(X) = \frac{\mathrm{i}}{48\pi^3} \int_{D'} \mathrm{tr}(h^{-1}\,\mathrm{d}h)^3\,\mathrm{d}X \tag{3.43}$$

for $h \in D'b$ and $X \in D'(\mathrm{Lie}\, G)$. We have

$$\theta_{U_{g'}} = \theta_{U_{f'}} + (\chi_{U_{f'}, U_{g'}})^{-1}\,\mathrm{d}\chi_{U_{f'}, U_{g'}},$$

where $\chi_{U_{f'}, U_{g'}}$ is the transition function of $\mathrm{WZ}(S^3)$

$$\chi_{U_{f'}, U_{g'}}(b) = \chi'(h', k')$$

for $h' \in D'b \cap V_{f'}$ and $k' \in D'b \cap V_{g'}$. We have a well-defined connection θ on $\mathrm{WZ}(S^3)$. The curvature of θ becomes

$$F(X, Y) = -\frac{1}{24\pi^3} \int_{S^3} \mathrm{tr}(V^2(X\,\mathrm{d}Y - Y\,\mathrm{d}X)), \quad V = \mathrm{d}f\, f^{-1}|S^3. \tag{3.44}$$

The calculation for these formula is the same as in [5,9,10].

Similarly we have a connection on $\mathrm{WZ}((S^3)')$ represented by a formula parallel to (3.43) but integrated on D.

On the pullback bundle $r^* \mathrm{WZ}(S^3)$ there is an induced covariant derivative

$$(r^* \triangledown)_X s(f) = (\triangledown_{r_* X} r_* s)(r(f)),$$

where $r_* s$ is the section of $\mathrm{WZ}(S^3)$ defined by $r_* s(b) = s(f) = [f', c'] \in \mathrm{WZ}(S^3)_b$ for an (and any) $f \in Db$. X is a vector field on D, hence $r_* X$ is a vector field on S^3.

Similarly the covariant derivative on $\mathrm{WZ}((S^3)')$ is defined.

The sections $\mathrm{WZ}(D)$ and $\mathrm{WZ}(D')$ are parallel with respect to the respective covariant derivation. This follows almost from the definitions by virtue of the infinitesimal form of the Polyakov–Wiegmann formula:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} C_5(f\,\mathrm{e}^{tX}) = \frac{\mathrm{i}}{48\pi^3} \int_{S^4} \mathrm{tr}(f^{-1}\mathrm{d}f)^3\,\mathrm{d}X \quad \text{for } X \in S^4(\mathrm{Lie}\, G), \quad f \in S^4 G. \tag{3.45}$$

Proposition 3.3.

$$\triangledown \mathrm{WZ}(D) = 0, \tag{3.46}$$

$$\triangledown \mathrm{WZ}(D') = 0. \tag{3.47}$$

Remark 3.3. We could consider in the following construction of the WZW model those line bundles $\mathrm{WZ}_n(S^3)$ associated to the nth sector of $\Omega^3 G$, but for a fixed n. However, in the sequel we shall restrict our discussion only to the contractible component $\Omega^3_0 G$.

4. Construction of WZW actions

4.1. *Line bundle* $\mathrm{WZ}(\Gamma)$

Let $\Sigma \in \mathcal{M}_4$. Then Σ is a conformally flat manifold with boundary $\partial\Sigma = \Gamma = \cup_{i\in I_+}\Gamma_i \cup \cup_{i\in I_-}\Gamma_i$ with Γ_i a parameterized round S^3 in Σ.

For an $i \in I_- \oplus I_+$, the parameterization defines the map $p_i : S^3 \to \Gamma_i$, and the map $p_i : \Gamma_i G \to \Omega^3 G$, which we denote by the same letter. Then we have the pullback bundle of $\mathrm{WZ}(S^3)$ (resp. $\mathrm{WZ}((S^3)'))$ by p_i. We define

$$\mathrm{WZ}(\Gamma_i) = p_i^* \,\mathrm{WZ}(S^3) \quad \text{for} \quad i \in I_-, \qquad \mathrm{WZ}(\Gamma_i) = p_i^* \,\mathrm{WZ}((S^3)') \quad \text{for} \quad i \in I_+, \tag{4.1}$$

then we have, respectively,

$$\mathrm{WZ}(\Gamma_i') = p_i^* \,\mathrm{WZ}((S^3)') \quad \text{for} \quad i \in I_-, \qquad \mathrm{WZ}(\Gamma_i') = p_i^* \,\mathrm{WZ}(S^3) \quad \text{for} \quad i \in I_+. \tag{4.2}$$

The line bundle $\mathrm{WZ}(\Gamma)$ is defined by

$$\mathrm{WZ}(\Gamma) = \bigotimes_{i\in I_-} \mathrm{WZ}(\Gamma_i) \otimes \bigotimes_{i\in I_+} \mathrm{WZ}(\Gamma_i). \tag{4.3}$$

Now let $\alpha : S^3 \to S^3$ be the restriction on S^3 of a conformal diffeomorphism on S^4. First we suppose that α preserves the orientation. Then, since the transition function χ is invariant under α, the line bundle $\mathrm{WZ}(S^3)$ is invariant under α. If α reverses the orientation then D is mapped to D' and χ is changed to χ'. Then $\alpha^*\,\mathrm{WZ}(S^3)$ becomes $\mathrm{WZ}((S^3)')$. On the other hand, the parameterizations p_i are uniquely defined up to composition with conformal diffeomorphisms. Therefore $\mathrm{WZ}(\Gamma)$ is well defined for the conformal equivalence class of $\Gamma \in \mathcal{M}$.

The dual of $\mathrm{WZ}(\Gamma)$ is

$$\mathrm{WZ}(\Gamma') = \bigotimes_{i\in I_-} \mathrm{WZ}(\Gamma_i') \otimes \bigotimes_{i\in I_+} \mathrm{WZ}(\Gamma_i'), \tag{4.4}$$

and the duality; $\mathrm{WZ}(\Gamma) \times \mathrm{WZ}(\Gamma') \to \mathrm{C}$, is given from (3.29) by

$$\left\langle \bigotimes_{i\in I_-} [f_i', c_i'] \otimes \bigotimes_{i\in I_+} [g_i, d_i], \bigotimes_{i\in I_-} [f_i, c_i] \otimes \bigotimes_{i\in I_+} [g_i', d_i'] \right\rangle$$
$$= \Pi_{i\in I_-} c_i c_i' \cdot \Pi_{i\in I_+} d_i d_i' \exp\left\{ -2\pi \mathrm{i} \sum_{i\in I_-} C_5(f_i \vee f_i') - 2\pi \mathrm{i} \sum_{i\in I_+} C_5(g_i \vee g_i') \right\}.$$

The above defined $\mathrm{WZ}(\Gamma)$ satisfies axioms A1 and A2.

4.2. *Non-vanishing section* $\mathrm{WZ}(\Sigma)$

In the following, we shall define step by step the section $\mathrm{WZ}(\Sigma)$ of $r^*\,\mathrm{WZ}(\Gamma)$ for any $\Sigma \in \mathcal{M}_4$ with the boundary $\partial\Sigma = \Gamma$ and $r : \Sigma G \to \Gamma G$.

We obtain a compact manifold $\Sigma^c \in \mathcal{M}_4$ without boundary by sewing a copy D_i of D along Γ_i for $i \in I_-$ and a copy D'_i of D' for $i \in I_+$

$$\Sigma^c = \left(\bigcup_{i\in I_-} D_i\right) \bigcup_{\cup_{I_-}\Gamma_i} \Sigma \bigcup_{\cup_{I_+}\Gamma_i} \left(\bigcup_{i\in I_+} D'_i\right).$$

For each boundary component Γ_i of Γ the parameterization p_i is extended to a parameterization $\tilde{p}_i : D_i \to D$ if $i \in I_-$, and $\tilde{p}_i : D_i \to D'$ if $i \in I_+$. The extension is unique up to composition with conformal transformations, see Section 2.1.

We put

$$\mathrm{WZ}(D_i) = (\tilde{p}_i)^*\,\mathrm{WZ}(D), \tag{4.5}$$

$$\mathrm{WZ}(D'_i) = (\tilde{p}_i)^*\,\mathrm{WZ}(D'). \tag{4.6}$$

For $i \in I_-$, $\mathrm{WZ}(D_i)$ is a section of the pullback bundle of $\mathrm{WZ}(\Gamma_i)$ by the restriction map $r_i : D_i G \to \Gamma_i G$, and $\mathrm{WZ}(D'_i)$ is a section of the pullback bundle of $\mathrm{WZ}(\Gamma'_i)$ by the restriction map $r'_i : D'_i G \to \Gamma_i G$. Similarly, for $i \in I_+$, $\mathrm{WZ}(D'_i)$ defines a section of the pullback line bundle of $\mathrm{WZ}(\Gamma_i)$ by r'_i, and $\mathrm{WZ}(D_i)$ is a section of $r_i^*\,\mathrm{WZ}(\Gamma'_i)$.

1. Let $\Sigma_1 \in \mathcal{M}_4$ and suppose that the compactified space $(\Sigma_1)^c$ is simply connected, that is, Σ_1 is a subset of S^4 deleted several discs D_i; $i \in I_-$ and D'_i; $i \in I_+$ with parameterized boundaries $\Gamma = \cup_{i\in I_-}\Gamma_i \cup \cup_{i\in I_+}\Gamma_i$. Let

$$\Phi_1 = \bigotimes_{i\in I_+} \mathrm{WZ}(D_i) \otimes \bigotimes_{i\in I_-} \mathrm{WZ}(D'_i), \tag{4.7}$$

Φ_1 is a section of the pullback bundle of $\mathrm{WZ}(\Gamma')$ by the restriction map

$$\left(\bigcup_{i\in I_-} D'_i \cup \bigcup_{i\in I_+} D_i\right) G \to \left(\bigcup_{i\in I_-} \Gamma_i \cup \bigcup_{i\in I_+} \Gamma_i\right) G.$$

Then $\mathrm{WZ}(\Sigma_1)$ is defined by the duality relation

$$\langle \mathrm{WZ}(\Sigma_1), \Phi_1\rangle = \mathrm{WZ}(S^4) = 1. \tag{4.8}$$

In fact, given $f \in \Sigma_1 G$, take $f_i \in D_i G$, $i \in I_+$, and $f'_i \in D'_i G$, $i \in I_-$, in such a way that $f|\Gamma_i = f_i|\Gamma_i, i \in I_+$, and $f|\Gamma_i = f'_i|\Gamma_i, i \in I_-$. Let $\mathrm{WZ}(D_i)(f_i) = (f_i, u_i), i \in I_+$, and $\mathrm{WZ}(D'_j)(f'_j) = (f'_j, u'_j)$, $j \in I_-$. By the definition

$$u_i \in \mathrm{WZ}(\Gamma'_i)_{r_i(f_i)} \quad \text{and} \quad u'_j \in \mathrm{WZ}(\Gamma'_j)_{r'_j(f'_j)}.$$

Then $\Phi_1((f_i)_{i\in I_+}, (f'_j)_{j\in I_-}) = ((f_i)_{i\in I_+}, (f'_j)_{j\in I_-}, \otimes_{i\in I_+} u_i \otimes \otimes_{j\in I_-} u'_j)$. There is a

$$v \in \bigotimes_{i\in I_+} \mathrm{WZ}(\Gamma_i)_{r_i(f_i)} \otimes \bigotimes_{j\in I_-} \mathrm{WZ}(\Gamma_j)_{r'_j(f'_j)} = \mathrm{WZ}(\Gamma)_{r(f)},$$

such that $\langle v, \otimes_{i\in I_+} u_i \otimes \otimes_{j\in I_-} u'_j\rangle = 1$. The definition of $\mathrm{WZ}(D_i)$ and $\mathrm{WZ}(D'_i)$ imply that v is independent of $\{f_i,\ f'_i\}$, but depends only on f.

Thus $\mathrm{WZ}(\Sigma_1)(f) = (f, v)$ is well defined as a section of the pullback bundle of $\mathrm{WZ}(\Gamma)$ by $r : \Sigma_1 G \to \Gamma G$.

2. Let $\Sigma_0 = S^3 \times [0, 1]$. We define

$$\mathrm{WZ}(\Sigma_0) = 1_{\mathrm{WZ}((S^3)')\otimes \mathrm{WZ}(S^3)}. \tag{4.9}$$

Then we have

$$\langle \mathrm{WZ}(\Sigma_0), \mathrm{WZ}(D) \otimes \mathrm{WZ}(D')\rangle = \langle \mathrm{WZ}(D), \mathrm{WZ}(D')\rangle = 1.$$

This is concordant with the definition in Step (1).

3. We shall call a $\Sigma_1 \in \mathcal{M}_4$ described in (1) that is not of cylinder type a *basic component*. Any $\Sigma \in \mathcal{M}_4$ can be decomposed to a sum of several basic components that are patched together by their parameterized boundaries:

$$\Sigma = \bigcup_{k=1}^{N} \Sigma_k. \tag{4.10}$$

The incoming boundaries of Σ_k coincide, respectively, with the outgoing boundaries of Σ_{k-1} up to their orientations, that is, $\Gamma_i^{k-1} = (\Gamma_i^k)'$, and Σ is obtained by patching together these boundaries. Then there is a duality of $\mathrm{WZ}(\Gamma_i^{k-1}) = (p_i^{k-1})^* \mathrm{WZ}((S^3)')$ and $\mathrm{WZ}(\Gamma_i^k) = (p_i^k)^* \mathrm{WZ}(S^3)$. Using a suitable Morse function on Σ, we may suppose that the parameterized boundaries Γ_i; $i \in I_-$ of Σ are all contained in the boundary $\partial\Sigma_1$ and Γ_i; $i \in I_+$ are in $\partial\Sigma_N$. Then we define

$$\mathrm{WZ}(\Sigma_2 \cup \Sigma_1) = \langle \mathrm{WZ}(\Sigma_2), \mathrm{WZ}(\Sigma_1)\rangle. \tag{4.11}$$

Here $\langle\cdot,\cdot\rangle$ is the natural pairing (contraction) between the line bundles $\otimes_{i\in I_-}\mathrm{WZ}(\Gamma_i) \otimes \otimes_{j\in J_+^1}\mathrm{WZ}(\Gamma_j)$ and $\otimes_{j\in J_-^2}\mathrm{WZ}(\Gamma_j) \otimes \otimes_{k\in J_+^2}\mathrm{WZ}(\Gamma_k)$. Here we have written $\partial\Sigma_1 = \cup_{j\in J_+^1}\Gamma_j \cup \cup_{i\in I_-}\Gamma_i$ and $\partial\Sigma_2 = \cup_{k\in J_+^2}\Gamma_k \cup \cup_{j\in J_-^2}\Gamma_j$, hence $\mathrm{WZ}(\Sigma_2 \cup \Sigma_1)$ is a section of the pullback line bundle of $\otimes_{i\in I_-}\mathrm{WZ}(\Gamma_i) \otimes \otimes_{k\in J_+^2}\mathrm{WZ}(\Gamma_k)$ by the boundary restriction map

$$r : (\Sigma_2 \cup \Sigma_1)G \to \bigcup_{i\in I_-}\Gamma_i G \cup \bigcup_{k\in J_+^2}\Gamma_k G,$$

see the explanation after A3 of Section 2.2.

Lemma 4.1. *Let* $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$. *Let their boundaries be*

$$\partial\Sigma_1 = \gamma_1 \cup \Gamma'_2 \cup \Gamma_3, \qquad \partial\Sigma_2 = \gamma_2 \cup \Gamma'_3 \cup \Gamma_1 \quad \text{and} \quad \partial\Sigma_3 = \gamma_3 \cup \Gamma'_1 \cup \Gamma_2.$$

Then we have

$$\langle\langle \mathrm{WZ}(\Sigma_1), \mathrm{WZ}(\Sigma_2)\rangle, \mathrm{WZ}(\Sigma_3)\rangle = \langle \mathrm{WZ}(\Sigma_1), \langle \mathrm{WZ}(\Sigma_2), \mathrm{WZ}(\Sigma_3)\rangle\rangle. \tag{4.12}$$

This is merely the problem of forming a tensor product of several line bundles, that is a commutative operation.

By virtue of this lemma we can form successively

$$\mathrm{WZ}(\Sigma_k \cup \Sigma_{(k-1)} \cup \cdots \cup \Sigma_1) = \langle \mathrm{WZ}(\Sigma_k), \langle \mathrm{WZ}(\Sigma_{(k-1)}), \ldots, \mathrm{WZ}(\Sigma_1)\rangle \cdots \rangle. \tag{4.13}$$

This is independent of the order of partition and is also independent of how to decompose $\Sigma^{(k)} = \Sigma_k \cup \Sigma_{(k-1)} \cup \cdots \cup \Sigma_1$, but depends only on $\Sigma^{(k)}$. Therefore

$$\mathrm{WZ}(\Sigma) = \mathrm{WZ}(\Sigma_N \cup \Sigma_{(N-1)} \cup \cdots \cup \Sigma_1)$$

is well defined as a section of the pullback line bundle of $\otimes_{i\in I_-}\mathrm{WZ}(\Gamma_i) \otimes \otimes_{i\in I_+}\mathrm{WZ}(\Gamma_i)$ by the boundary restriction map.

From the construction, $\mathrm{WZ}(\Sigma)$ satisfies axiom A3.

Now let $\Sigma \in \mathcal{M}_4$ be compact without boundary. Let Σ_1 and Σ_2 be the basic components such that $\Sigma = \Sigma_1 \cup_\Gamma \Sigma_2$. Suppose that

$$\partial\Sigma_1 = \Gamma = \bigcup_{i\in I}\Gamma_i', \qquad \partial\Sigma_2 = \Gamma = \bigcup_{i\in I}\Gamma_i.$$

Then from the definition of $\mathrm{WZ}(\Sigma_i)$, $i = 1, 2$, we see that

$$\mathrm{WZ}(\Sigma) = \langle \mathrm{WZ}(\Sigma_2), \mathrm{WZ}(\Sigma_1)\rangle = \left\langle \bigotimes_{i\in I}\mathrm{WZ}(D_i'), \bigotimes_{i\in I}\mathrm{WZ}(D_i)\right\rangle = \sum_{i\in I} 1.$$

Thus we have the following proposition.

Proposition 4.1. *For any $\Sigma \in \mathcal{M}_4$ which is compact without boundary* $\mathrm{WZ}(\Sigma)$ *is a positive integer*.

Proposition 4.2. *Let $\Sigma \in \mathcal{M}_4$ and let Σ^{ij} be obtained from Σ by identifying the boundaries Γ_i, $i \in I_-$, and Γ_j, $j \in I_+$, via $p_j \cdot (p_i)^{-1} : \Gamma_i \to \Gamma_j$. Then*

$$\mathrm{WZ}(\Sigma^{ij}) = \mathrm{tr}_{ij}\,\mathrm{WZ}(\Sigma), \tag{4.14}$$

where tr_{ij} *are the trace maps* (*contraction*) *between* $r^*\,\mathrm{WZ}(\Gamma_i')$ *and* $r^*\,\mathrm{WZ}(\Gamma_j)$ *in the tensor product* $\otimes_{k\in I_-}\mathrm{WZ}(\Gamma_k') \otimes \otimes_{l\in I_+}\mathrm{WZ}(\Gamma_l)$.

Connections on $\mathrm{WZ}(\Gamma_i)$ and $\mathrm{WZ}(\Gamma_i')$ are defined naturally as the induced connections by (4.1) and (4.2). Obviously, $\mathrm{WZ}(D_i)$ and $\mathrm{WZ}(D_i')$ are parallel with respect to these connections. By the formulas of definitions (4.3), (4.8) and (4.13) we have a naturally induced connection on $\mathrm{WZ}(\Gamma)$ with respect to which $\mathrm{WZ}(\Sigma)$ is parallel. Therefore axiom A4 is verified.

Remark 4.1. Let $\Sigma \in \mathcal{M}_4$ and the boundary $\Gamma = \partial\Sigma$ be such that $\Gamma = \cup_{i\in I_+}\Gamma_i \cup \cup_{i\in I_-}\Gamma_i'$ with Γ_i a parameterized round S^3. Let $r_\pm$ denote, respectively, the restriction maps onto $\otimes_{i\in I_\pm}(\Gamma_i G)$. Then

$$\mathrm{WZ}(\Sigma) : r_-^*\left(\bigotimes_{i\in I_-}\mathrm{WZ}(\Gamma_i)\right) \to r_+^*\left(\bigotimes_{i\in I_+}\mathrm{WZ}(\Gamma_i)\right),$$

$\mathrm{WZ}(\Sigma)(f)$ for $f \in \Sigma G$ is the higher-dimensional parallel transport along the "path" f [17]. When $I_+ = \phi$ or $I_- = \phi$ we call $\mathrm{WZ}(\Sigma)(f)$ the higher-dimensional holonomy along f.

4.3. Polyakov–Wiegmann formula

To see that the functor WZ satisfies the axioms of WZW model it remains for us to verify the axiom A5, the Polyakov–Wiegmann formula on every $\Sigma \in \mathcal{M}_4$. We have already seen the Polyakov–Wiegmann formula on S^4G, DG and $D'G$ in (3.17), (3.37) and (3.40), respectively.

Let $\Sigma \in \mathcal{M}_4$ with parameterized boundaries $\Gamma = \cup_{i\in I_-}\Gamma_i \cup \cup_{j\in I_+}\Gamma_j$. We shall use the same notation as in Sections 4.1 and 4.2. Then the product on each pullback line bundle $r_i^*\,\mathrm{WZ}(\Gamma_i)$, $(r')_i^*\,\mathrm{WZ}(\Gamma_i)$, $r_i^*\,\mathrm{WZ}(\Gamma_i')$ and $(r')_i^*\,\mathrm{WZ}(\Gamma_i')$ is defined in an obvious manner, and the non-vanishing sections $\mathrm{WZ}(D_i)$ and $\mathrm{WZ}(D_i')$ for $i \in I_\pm$ satisfy the respective Polyakov–Wiegmann formula

$$\mathrm{WZ}(D_i)(fg) = \mathrm{WZ}(D_i)(f) * \mathrm{WZ}(D_i)(g), \text{ etc.}$$

The products on the line bundle $S = \otimes_{j\in I_-}(r_j')^*\,\mathrm{WZ}(\Gamma_j) \otimes \otimes_{i\in I_+}(r_i)^*\,\mathrm{WZ}(\Gamma_i)$ and on the line bundle $S^* = \otimes_{j\in I_-}(r_j')^*\,\mathrm{WZ}(\Gamma_j') \otimes \otimes_{i\in I_+}(r_i)^*\,\mathrm{WZ}(\Gamma_i')$ are defined by tensoring the product on each $r_i^*\,\mathrm{WZ}(\Gamma_i)$, etc. We note also that the products are compatible with the duality

$$\langle \alpha * \beta, \lambda * \mu\rangle = \langle \alpha, \lambda\rangle * \langle \beta, \mu\rangle$$

for $\alpha, \beta \in S$ and $\lambda, \mu \in S^*$. Where the product in the right-hand side is that in $\mathrm{WZ}(\phi) \simeq \mathrm{C}$.

Now suppose that Σ is a subset of S^4 deleted several discs D_i, $i \in I_\pm$. Let $r : \Sigma G \to \Gamma G$ be the restriction map. Then the product on $r^*\,\mathrm{WZ}(\Gamma)$ is derived from the product on S. In fact, if we write $r(f) = (r_i(f_i); i \in I_+, r_j'(f_j'); j \in I_-)$ as in the argument of Section 4.2, then $\mathrm{WZ}(\Gamma)_{r(f)} = S_{r_j'(f_j'), r_i(f_i)}$, so the product on S yields that on $r^*\,\mathrm{WZ}(\Gamma)$, which is seen to be independent of the choice of $\{f_i, f_j'\}$.

Let $\Phi_1 = \otimes_{i\in I_+}\mathrm{WZ}(D_i) \otimes \otimes_{j\in I_-}\mathrm{WZ}(D_j')$. Φ_1 is a section of the line bundle S^* and satisfies

$$\Phi_1(f'g') = \Phi_1(f') * \Phi_1(g')$$

for $f', g' \in \otimes_{i\in I_+}D_iG \otimes \otimes_{i\in I_-}D_i'G$. Since the section $\mathrm{WZ}(\Sigma)$ of $r^*\,\mathrm{WZ}(\Gamma)$ was defined by the duality; $\langle \mathrm{WZ}(\Sigma), \Phi_1\rangle = \mathrm{WZ}(S^4)$, we have

$$\begin{aligned}\langle \mathrm{WZ}(\Sigma)(fg), \Phi_1(f'g')\rangle &= \mathrm{WZ}(S^4)(fg \vee f'g') = \mathrm{WZ}(S^4)(f \vee f') * \mathrm{WZ}(S^4)(g \vee g') \\ &= \langle \mathrm{WZ}(\Sigma)(f), \Phi_1(f')\rangle * \langle \mathrm{WZ}(\Sigma)(g), \Phi_1(g')\rangle \\ &= \langle \mathrm{WZ}(\Sigma)(f) * \mathrm{WZ}(\Sigma)(g), \Phi_1(f'g')\rangle\end{aligned}$$

for any $f, g \in \Sigma G$ and for f' and g' that are extensions of f and g to $\cup_{i\in I_-}D_i' \cup \cup_{j\in I_+}D_j$, respectively. Therefore we have

$$\mathrm{WZ}(\Sigma)(fg) = \mathrm{WZ}(\Sigma)(f) * \mathrm{WZ}(\Sigma)(g)$$

for $f, g \in \Sigma G$.

Let $\Sigma = \Sigma_1 \cup_\Gamma \Sigma_2$. The product operations on $(r_i)^* \mathrm{WZ}(\Gamma_i)$, $i = 1, 2$, are compatible with the contraction, in particular we have

$$\langle \mathrm{WZ}(\Sigma_1)(f_1) * \mathrm{WZ}(\Sigma_1)(g_1), \mathrm{WZ}(\Sigma_2)(f_2) * \mathrm{WZ}(\Sigma_2)(g_2)\rangle \\ = \mathrm{WZ}(\Sigma)(f) * \mathrm{WZ}(\Sigma)(g), \tag{4.15}$$

where $f, g \in \Sigma G$ and $f_i = f|\Sigma_1$, $i = 1, 2$, etc. For a general $\Sigma \in \mathcal{M}_4$ the formula follows from (4.15) and the definition of $\mathrm{WZ}(\Sigma)$ in (4.13). Thus we have proven the following generalization of the Polyakov–Wiegmann formula.

Theorem 4.1.

$$\mathrm{WZ}(\Sigma)(f) * \mathrm{WZ}(\Sigma)(g) = \mathrm{WZ}(\Sigma)(fg) \tag{4.16}$$

for $f, g \in \Sigma G$.

5. Extensions of the group $\Omega_0^3 G$

It is a well-known observation that the two-dimensional WZW action gives a geometric description of the central extension $\widehat{LG}$ of the loop group LG. The associated group cocycle yields a Lie algebra cocycle for the affine Kac–Moody algebra based on $\mathrm{Lie}(G)$ [2,6]. The total space of the $U(1)$-principal bundle $\widehat{LG}$ was described as the set of equivalence classes of pairs $(f, c) \in D^2G \times U(1)$, where D^2 is the two-dimensional disc with boundary S^1. The equivalence relation was defined on the basis of Polyakov–Wiegmann formula [12], as it was so in our four-dimensional generalization treated in Section 3.

Associated to the line bundle $\mathrm{WZ}(S^3)$ there exists a $U(1)$-principal bundle over $\Omega_0^3 G$. However, this bundle has not any natural group structure contrary to the case of the extension of loop group. Instead, Mickelsson [10] gave an extension of $\Omega_0^3 G$ by the Abelian group $\mathrm{Map}(\mathcal{A}_3, U(1))$, where $\mathcal{A}_3$ is the space of connections on S^3. In the following, we shall explain after [9] two extensions of $\Omega_0^3 G$ by the Abelian group $\mathrm{Map}(\mathcal{A}_3, U(1))$ that are in duality.

5.1. Mickelsson's 2-cocycle

We consider the quotient space

$$\widehat{\Omega G} = D'G \times \mathrm{Map}(\mathcal{A}_3, U(1))/\sim', \tag{5.1}$$

where $\sim'$ is the equivalence relation defined by

$$(f', \lambda) \sim' (g', \mu) \quad \text{if and only if} \quad f'|S^3 = g'|S^3, \\ \mu(A) = \lambda(A)\chi'(f', g') \quad \text{for any} \quad A \in \mathcal{A}_3. \tag{5.2}$$

The projection $\pi : \widehat{\Omega G} \to \Omega_0^3 G$ is defined by $\pi([f', \lambda]) = f'|S^3$. Then $\widehat{\Omega G}$ becomes a principal bundle over $\Omega_0^3 G$ with the structure group $\mathrm{Map}(\mathcal{A}_3, U(1))$. Here the $U(1)$ valued transition function $\chi'(f', g')$ is considered as a constant function in $\mathrm{Map}(\mathcal{A}_3, U(1))$.

The group structure of $\widehat{\Omega G}$ is given by the Mickelsson's 2-cocycle (3.9) on D':

$$\gamma_{D'}(\cdot;\, f', g') \quad \text{for} \quad f', g' \in D'G.$$

We note that since it is the coboundary of

$$\frac{\mathrm{i}}{24\pi^3}\int_{D'} \alpha_4(A;\, f'),$$

$\gamma_{D'}$ is in fact a cocycle. We define the product on $D'G \times \mathrm{Map}(\mathcal{A}_3, U(1))$ by

$$(f', \lambda) * (g', \mu) = (f'g', \lambda(\cdot)\mu_{f'}(\cdot)\exp\{2\pi \mathrm{i}\gamma_{D'}(A;\, f', g')\}), \tag{5.3}$$

where

$$\mu_{f'}(A) = \mu((f'|S^3)^{-1}A(f'|S^3) + (f'|S^3)^{-1}\,\mathrm{d}(f'|S^3)).$$

Then $D'G \times \mathrm{Map}(\mathcal{A}_3, U(1))$ is endowed with a group structure and $\widehat{\Omega G}$ inherits it. The group $\mathrm{Map}(\mathcal{A}_3, U(1))$ is embedded as a normal subgroup in $\widehat{\Omega G}$. Thus $\widehat{\Omega G}$ is an extension of $\Omega_0^3 G$ by the Abelian group $\mathrm{Map}(\mathcal{A}_3, U(1))$ [9,10].

We have another extension of $\Omega_0^3 G$ by $\mathrm{Map}(\mathcal{A}_3, U(1))$ if we consider

$$\widehat{\Omega' G} = DG \times \mathrm{Map}(\mathcal{A}_3, U(1))/\sim, \tag{5.4}$$

where the equivalence relation $\sim$ is defined by

$$(f, \lambda) \sim (g, \mu) \quad \text{if and only if} \quad f|S^3 = g|S^3,$$
$$\mu(A) = \lambda(A)\chi(f, g) \quad \text{for any} \quad A \in \mathcal{A}_3. \tag{5.5}$$

The product on $\widehat{\Omega' G}$ is defined by the same way as above using the 2-cocycle $\gamma_D(A;\, f, g)$ of (3.9), and $\widehat{\Omega' G}$ becomes an extension of $\Omega_0^3 G$ by the Abelian group $\mathrm{Map}(\mathcal{A}_3, U(1))$.

The group $\mathrm{Map}(\mathcal{A}_3, U(1))$ acts on C by $\lambda \cdot c = \lambda(0)c$. Then the associated line bundle to $\widehat{\Omega G}$ is $\mathrm{WZ}(S^3)$ and that associated to $\widehat{\Omega' G}$ is $\mathrm{WZ}((S^3)')$.

Remark 5.1. Consider the empty three manifold ϕ and look it as the boundary of S^4. Then we may follow the above definition to have an extension of ϕG by $\mathrm{Map}(\mathcal{A}_3, U(1))$. It becomes $\widehat{\phi G} = S^4 G \times \mathrm{Map}(\mathcal{A}_3, U(1))/\sim$, where

$$(F, \lambda) \sim (G, \mu) \quad \text{if and only if} \quad \mu(A) = \exp\{2\pi \mathrm{i}\omega(F, F^{-1}G)\}\lambda(A) \quad \text{for any} \quad A.$$

Then, since $(F, \lambda) \sim (F, \lambda(0))$, it reduces to $\widehat{\phi G} = S^4 G \times U(1)/\sim$, that is, $\widehat{\phi G} \simeq U(1)$. The product in $\widehat{\phi G}$ may be defined by the same formula as in (5.3), but we have seen that it reduces to that of (3.16) because of the equality $\gamma_{S^4}(A;\, F, G) = \gamma(F, G)$, (3.10).

The duality between two extensions $\widehat{\Omega G}$ and $\widehat{\Omega' G}$ is given as follows. For $[f', \lambda] \in \widehat{\Omega G}$ and $[f, \alpha] \in \widehat{\Omega' G}$, we put

$$\langle [f', \lambda], [f, \alpha] \rangle = [f \vee f', \lambda(0)\alpha(0)], \tag{5.6}$$

where on the right-hand side we used the product in $\widehat{\phi G} \simeq U(1)$. In fact, suppose that $(f', \lambda) \sim' (g', \mu)$ and $(f, \alpha) \sim (g, \beta)$. Then we have

$$\mu(A)\beta(A) = \lambda(A)\alpha(A) \exp\{2\pi \mathrm{i}[\gamma_{D'}(A; f', g') + \gamma_D(A; f, g)]\}$$
$$= \lambda(A)\alpha(A) \exp\{2\pi \mathrm{i}\gamma(f \vee f', g \vee g')\}. \tag{5.7}$$

Here we used the relation (3.11).

The Lie algebra cocycle corresponding to the group cocycle γ_D is calculated in [10]. It is given by

$$c(A; X, Y) = \frac{\mathrm{i}}{12\pi^2} \int_D \mathrm{tr}\, \mathrm{d}A(\mathrm{d}X\, \mathrm{d}Y + \mathrm{d}Y\, \mathrm{d}X)$$
$$= \frac{\mathrm{i}}{12\pi^2} \int_{S^3} \mathrm{tr}(A(\mathrm{d}X\, \mathrm{d}Y + \mathrm{d}Y\, \mathrm{d}X)). \tag{5.8}$$

The Lie algebra cocycle corresponding to the group cocycle $\gamma_{D'}$ is given by

$$\frac{\mathrm{i}}{12\pi^2} \int_{D'} \mathrm{tr}\, \mathrm{d}A(\mathrm{d}X\, \mathrm{d}Y + \mathrm{d}Y\, \mathrm{d}X)$$
$$= -\frac{\mathrm{i}}{12\pi^2} \int_{S^3} \mathrm{tr}(A(\mathrm{d}X\, \mathrm{d}Y + \mathrm{d}Y\, \mathrm{d}X)) = -c(A; X, Y). \tag{5.9}$$

5.2. *Remarks*

1. The Euclidean action of a field $\varphi : \Sigma \to G$ in WZW conformal field theory is defined as

   $$S_\Sigma(\varphi) = -\frac{\mathrm{i}k}{12\pi^2} \int_\Sigma \mathrm{tr}(\mathrm{d}\varphi^{-1} \wedge *\mathrm{d}\varphi) + C_\Sigma(\varphi), \tag{5.10}$$

   $S_\Sigma(\varphi)$ is invariant under a conformal change of metric and the second term $C_\Sigma(\varphi)$ is required to obtain a conformal invariance of the action. This was shown by Fujii [5], and first noticed by Witten [18] for the two-dimensional WZW model. The kinetic term in (5.11) is linear with respect to the multiplication of the fields

   $$\int_\Sigma \mathrm{tr}(\mathrm{d}(fg)^{-1} \wedge *\mathrm{d}(fg)) = \int_\Sigma \mathrm{tr}(\mathrm{d}f^{-1} \wedge *\mathrm{d}f) + \int_\Sigma \mathrm{tr}(\mathrm{d}g^{-1} \wedge *\mathrm{d}g), \tag{5.11}$$

   and does not affect the Polyakov–Wiegmann formula. Hence we preferred only to deal with the topological term $C_\Sigma(f)$ [3].
2. The argument in this paper will be valid also for $2n$-dimensional conformally flat manifolds with boundary if the Lie group $G = \mathrm{SU}(N)$ is such that $N \geq n+1$, in this case we have $\pi_{2n}(G) = 0$ and $\pi_{2n+1}(G) = \mathbf{Z}$. We shall also have the Abelian extensions of $\Omega_0^{2n-1}G$ by $\mathrm{Map}(\mathcal{A}_{2n-1}, U(1))$. For that purpose we must have the Polyakov–Wiegmann formula for the action functional

   $$C_{2n+1}(f) = -\frac{\mathrm{i}}{(2n-1)!(2\pi \mathrm{i})^{2(n-1)}} \int_{D^{2n+1}} \mathrm{tr}(\tilde{g}^{-1}\, \mathrm{d}\tilde{g})^{2n+1}, \quad g \in S^{2n}G,$$

see [5]. It seems that Polyakov–Wiegmann formula has not yet been proved for general n larger than 3.

3. Losev et al. [8] discussed a four-dimensional WZW theory based on Kähler manifolds. Their Lagrangian is defined by

$$-\frac{1}{4\pi}\int_{\Sigma}\omega\wedge \mathrm{tr}(g^{-1}\partial g\wedge *g^{-1}\bar{\partial}g)+\frac{\mathrm{i}}{12\pi}\int_{\Sigma\times[0,1]}\omega\wedge \mathrm{tr}(g^{-1}\mathrm{d}g)^3.$$

The theory has the finiteness properties for the one-loop renormalization of the vacuum state. The authors studied the algebraic sector of their theory. The category of algebraic manifolds is not well behaved under contraction, hence their theory does not fit our axiomatic description.

4. S^4 is obtained by patching together two quaternion spaces and we have the conjugation $q\to q^{-1}$ on it. Under the conjugation $\mathrm{WZ}(S^4)$ is invariant but $\mathrm{WZ}(D)$ and $\mathrm{WZ}(D')$ will interchange. Since the conjugation inverts the orientation, $\mathrm{WZ}(\Sigma)$ is invariant under the conjugation of Σ. We can convince ourselves of this fact if we follow the argument to define $\mathrm{WZ}(\Sigma)$ for a $\Sigma\in\mathcal{M}_4$. This is the CPT invariance.

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